Robust Ellipsoid-specific Fitting via Expectation Maximization

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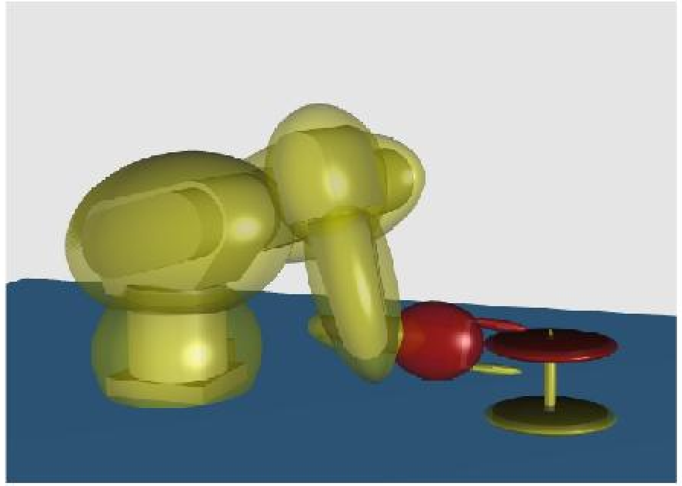
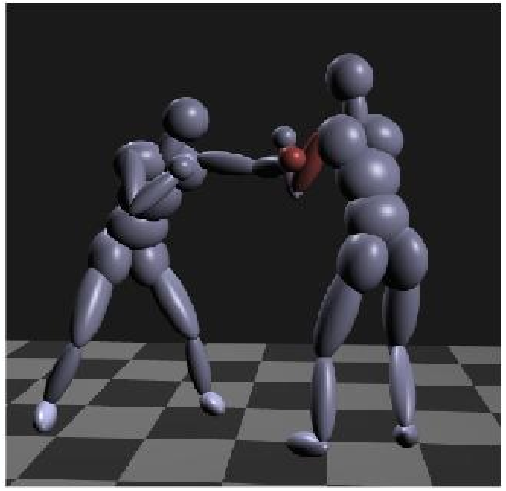
APPLICATIONS — ELLIPSOID FITTING

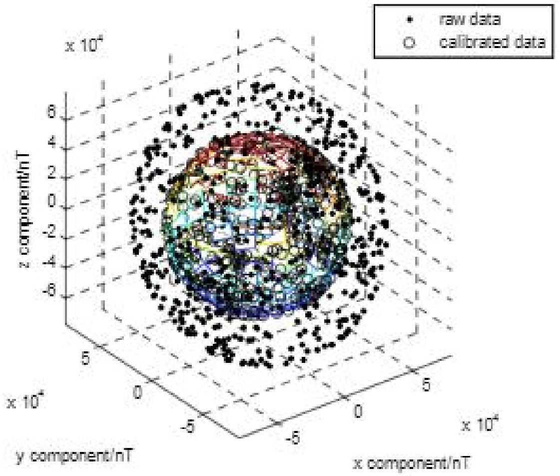
[Lu et al. 2007]



[Choi et al. 2009]



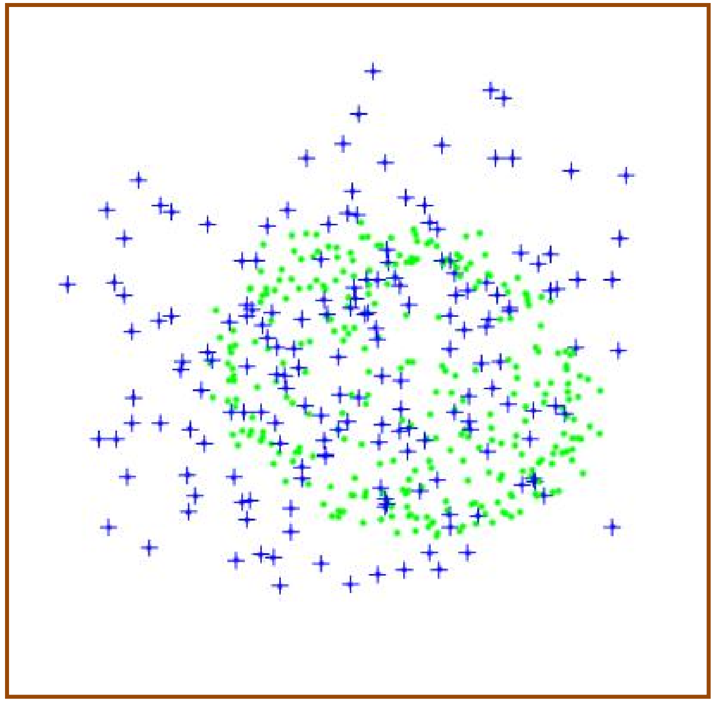
[Choi et al. 2009]



[Chi et al. 2018]

DEFINITION — ELLIPSOID FITTING

Our objective: Recognize the ellipsoid from noisy or outliers-contaminated data points



QUADRICS

$$Q(a,p) = a^T p = Ax^2 + By^2 + Cz^2 + 2Dxy + 2Exz^2 + 2Fyz + 2Gx + 2Hy + 2Iz + J = 0$$

$$a = [x^2, y^2, z^2, 2xy, 2xz, 2yz, 2x, 2y, 2z, 1]^T$$

$$p = [A, B, C, D, E, F, G, H, I, J]^T$$

TRADITIONAL METHODS

Algebra

$$\min \frac{1}{n}\sum_{i=1}^{n}(\mathbf{a}_{\mathrm{i}},\boldsymbol{\theta})^2$$

Easy to solve

Sensitive to outliers

VS.

Geometry

$$\min \frac{1}{n}\sum_{i=1}^{n}\mathrm{d}_i^{\,2}$$

More accurate

Sensitive to outliers

OUR METHOD — ELLIPSOID FITTING

BMVC 2021

Relative-density-based outlier score——RDOS

$$\mathrm{RDOS}(\mathbf{x}_i) = \frac{\sum_{\mathbf{x}\in \mathrm{N}(\mathbf{x}_i)} p(\mathbf{x})}{|\mathrm{N}(\mathbf{x}_i)|p(\mathbf{x}_i)}$$

$$p(\mathbf{x}_i) = \frac{1}{k+1} \sum_{\mathbf{x}\in \mathrm{N}(\mathbf{x}_i)\cup\{\mathbf{x}_i\}} \frac{1}{(2\pi h^2)^{d/2}} \exp(-\frac{\|\mathbf{x}-\mathbf{x}_i\|^2}{2h^2})$$

$$h = \frac{1}{k} \sum_{\mathbf{x}\in \mathrm{N}(\mathbf{x}_i)} (\mathbf{x}-\mathbf{x}_i)^T(\mathbf{x}-\mathbf{x}_i)$$

$$\left.\begin{array}{l} 0 < \mathrm{RDOS}(\mathbf{x}_i) < 1 \\ \mathrm{RDOS}(\mathbf{x}_i) \approx 1 \end{array}\right\} \mathbf{x}_i \text{ is not an outlier}$$

$\mathrm{RDOS}(\mathbf{x}_i) \gg 1 \Rightarrow \mathbf{x}_i$ is maybe an outlier

OUR METHOD — ELLIPSOID FITTING

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Create an unit spherical surface based on RDOS

$$\mathbf{Y} = \{\mathbf{y_m} = (x_m, y_m, z_m) \in \mathrm{R}^3\}_{m=1}^{M}$$

$$\begin{cases} x_m = x_c + \cos\theta_i \cdot \sin\psi_j \\ y_m = y_c + \cos\theta_i \cdot \cos\psi_j \\ z_m = z_c + \sin\theta_i \end{cases}$$

$$\theta_i = \frac{\pi i}{[\sqrt{M}]} \qquad \psi_j = \frac{2\pi j}{[\sqrt{M}]} \qquad (x_c, y_c, z_c) = \frac{1}{M}\sum_{\mathrm{RDOS}(\mathbf{x}_i)\leq 2} \mathbf{x}_i$$

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Use maximization likelihood for parameter estimation

$$p(\mathbf{z}) = \boxed{\frac{w}{V}} + \boxed{\frac{1-w}{M}\sum_{m=1}^{M}(2\pi\sigma^2)^{-D/2}\exp\{-\frac{1}{2\sigma^2}||\mathbf{z}-\mathbf{y}_m||^2\}}$$

w

Model outliers by the uniform distribution

Model the sphere by Gaussian Mixture Models

Probability distribution+data points——>maximization likelihood estimation （MLE）

Object: Minimize the negative log likelihood function

$$\min_{\Omega} E(\Omega \mid \mathbf{X}) = \min_{\Omega} - \sum_{i=1}^{N} \log p(x_i \mid \Omega)$$

$$= \min_{\Omega} - \sum_{i=1}^{N} \log \{ \sum_{m=1}^{M} \frac{1-w}{M} \frac{1}{(2\pi\sigma^2)^{d/2}} \exp\{ \frac{\| x_i - y_m \|^2}{2\sigma^2} \} + \frac{w}{V} \}$$

Solve for parameters by Expectation Maximization

E

Use the **Bayesian principle** to update the posterior probability

M

Update parameters by minimizing $Q(x)$

$$
\begin{aligned}
Q(\Omega,\Omega^{old}) &= \mathbf{E}_{\mathbf{Y}}[-\log p(\mathbf{Y},\mathbf{X}|\Omega)|\mathbf{X},\Omega^{old}] \\
&= -\Sigma_{\mathbf{Y}} \log p(\mathbf{Y},\mathbf{X}|\Omega) p(\mathbf{Y}|\mathbf{X},\Omega^{old}) \\
&= -\Sigma_{i=1}^{N}\Sigma_{m=1}^{M+1} p^{old}(\mathbf{y}_m|\mathbf{x}_n)\log(p^{new}(\mathbf{y}_m)p^{new}(\mathbf{x}_i|\mathbf{y}_m)) \\
&= \frac{1}{2\sigma^2}\Sigma_{i=1}^{N}\Sigma_{m=1}^{M} p^{old}(\mathbf{y}_m|\mathbf{x}_i,\Omega)\|\mathbf{x}_i-(\mathbf{A}\mathbf{y}_m+\mathbf{t})\|^2 \\
&+ \frac{N_p D}{2}\log\sigma^2 - \log(w)N_o - \log(1-w)N_p,
\end{aligned}
$$

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Accelerate EM convergence by the vector epsilon algorithm

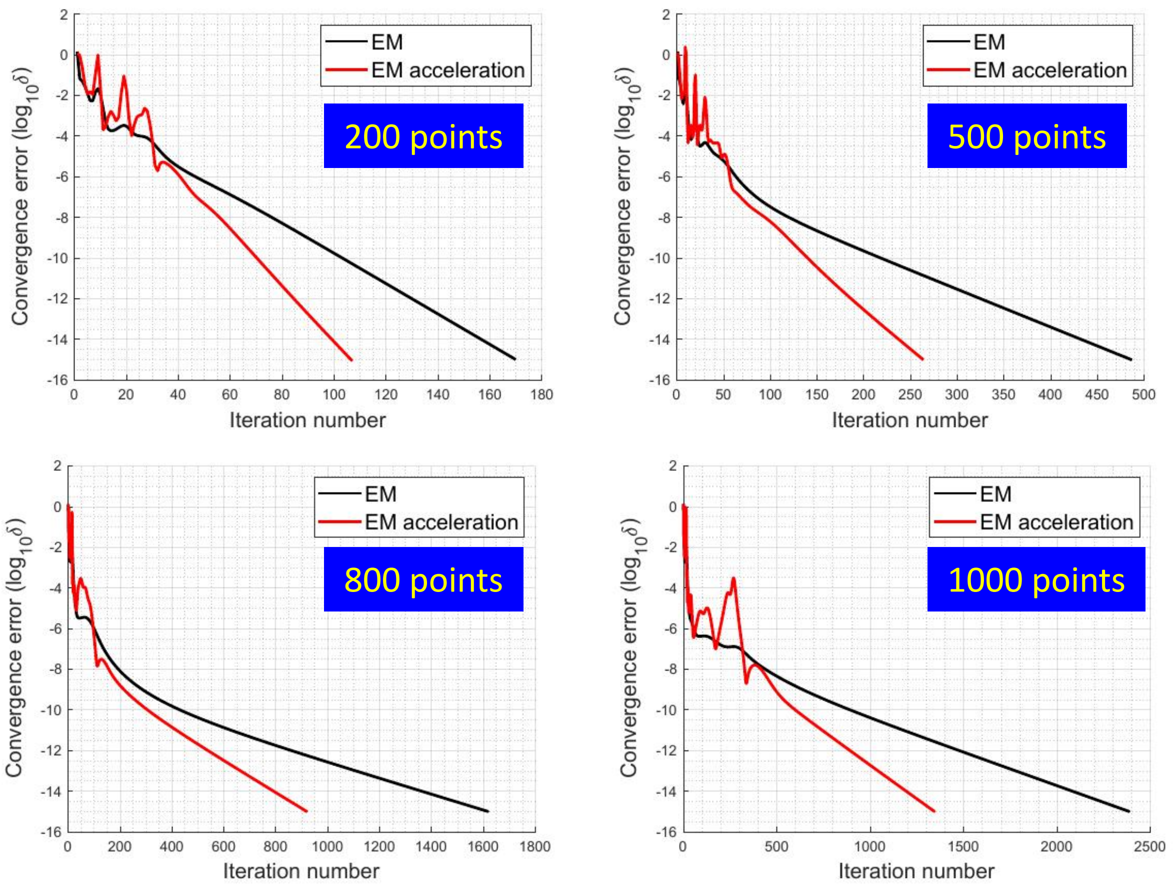
Theorem[1]: Suppose $\{\phi(t)\}_{t\geq 0}$ is the original sequence of EM, $\{\dot{\phi}(t)\}_{t\geq 0}$ is the new sequence generated by the vector epsilon EM, then

$$\lim_{t\to\infty}\frac{\|\dot{\phi}(t)-\overset{*}{\phi}\|}{\|\dot{\phi}(t+2)-\overset{*}{\phi}\|}=0$$

[1] Mingfeng Wang, Masahiro Kuroda, Michio Sakakihara, and Zhi Geng. Acceleration of the em algorithm using the vector epsilon algorithm. Computational Statistics, 23 (3):469–486, 2008.

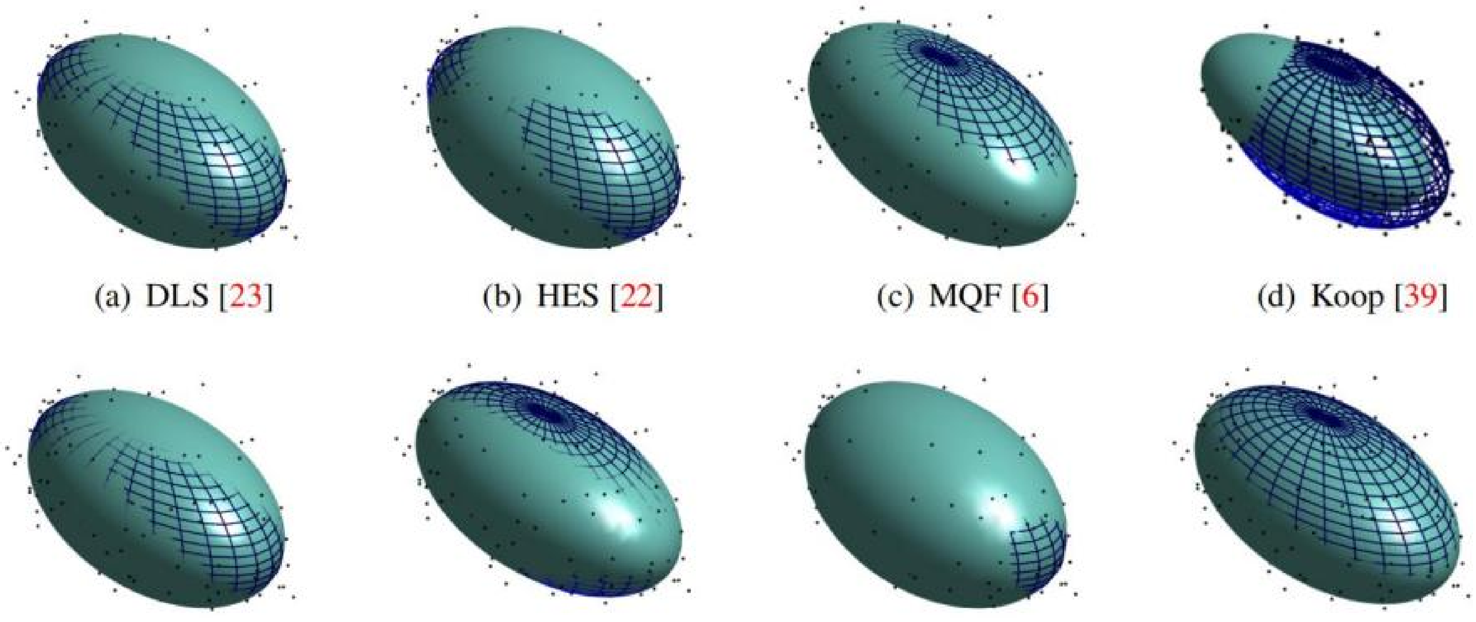
Experimental Evaluation

EXPERIMENTS— EM ACCELERATION



EM is much faster for more fitting points or with higher accuracy requirement

EXPERIMENTS— ROBUST AGAINST NOISE



(a) DLS [23] (b) HES [22] (c) MQF [6] (d) Koop [39]

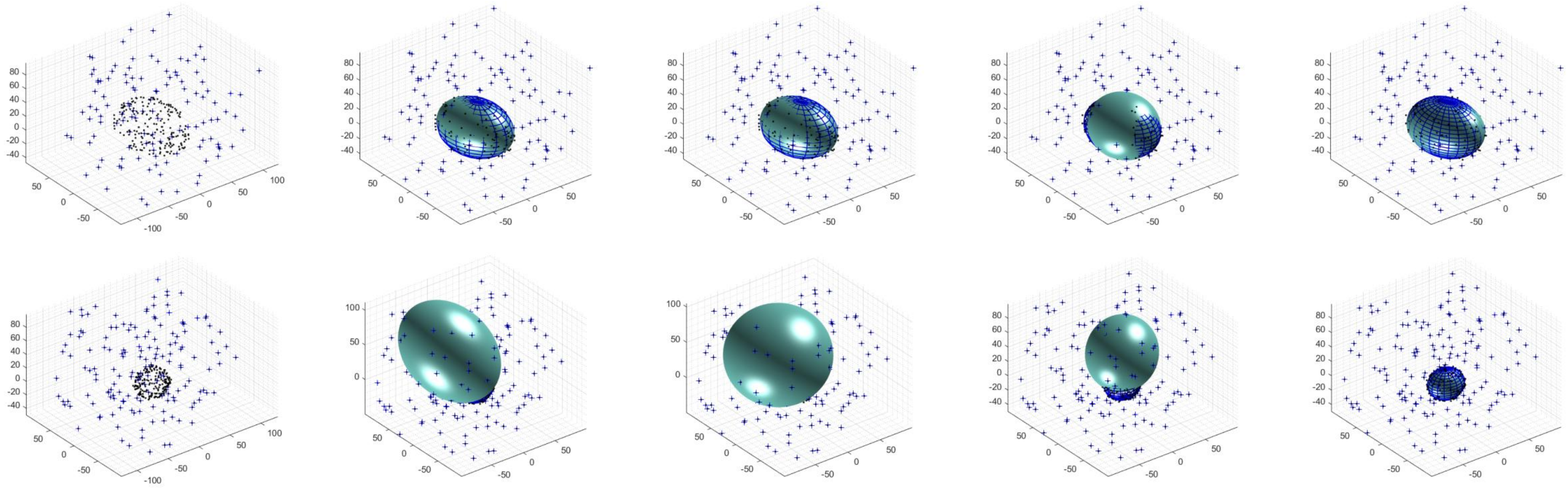
(e) Taubin [37] (f) GF [4] (g) RIX [25] (h) Ours

Table 1: Comparisons of different methods on noisy data, where bold font is the top fitter.

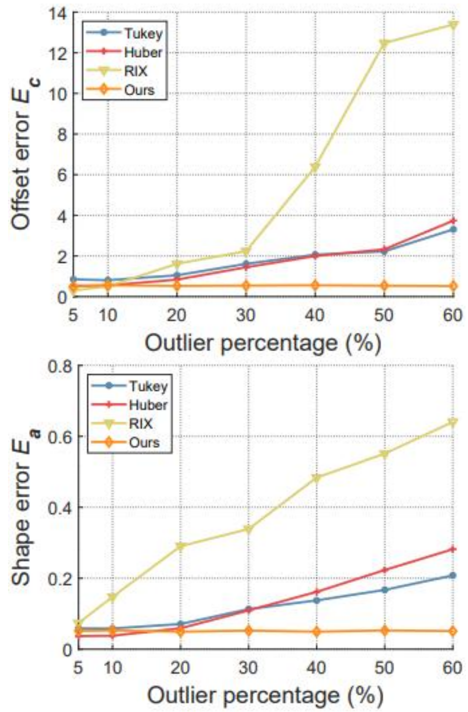
Noise (%) \ Method	Metric	DLS[22]	HES[21]	MQF[6]	Koop[38]	Taubin[36]	GF[3]	RIX[24]	Ours
5	E_c	3.43	3.42	3.47	4.06	3.89	1.31	**0.67**	1.03
	E_a	0.45	0.46	0.57	0.74	0.63	**0.14**	0.15	**0.14**
10	E_c	3.92	3.90	4.14	5.20	4.83	1.90	**1.17**	1.33
	E_a	0.47	0.48	0.65	0.88	0.71	0.21	0.23	**0.20**
15	E_c	4.51	4.49	4.11	6.12	5.87	2.14	1.66	**1.58**
	E_a	0.48	0.49	0.76	1.11	0.86	0.29	0.29	**0.25**
20	E_c	4.61	4.60	3.62	7.83	6.86	3.23	2.20	**2.01**
	E_a	0.46	0.47	0.84	1.43	0.94	0.53	0.33	**0.32**
25	E_c	4.85	4.84	5.10	8.97	8.81	4.21	2.68	**2.16**
	E_a	0.44	0.46	1.09	1.65	1.18	0.94	**0.38**	0.40

Our method is more robust against noise

EXPERIMENTS— ROBUST AGAINST OUTLIERS

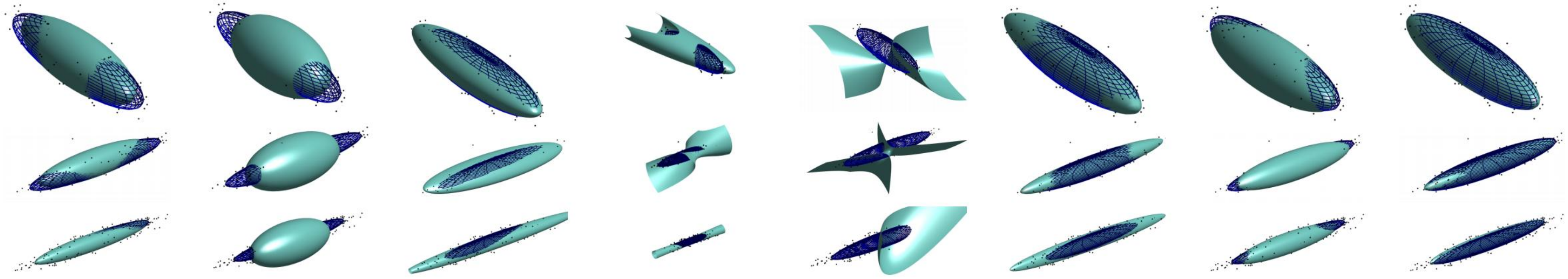


(a) Input (b) Tukey [32] (c) Huber [19] (d) RIX [25] (e) Ours

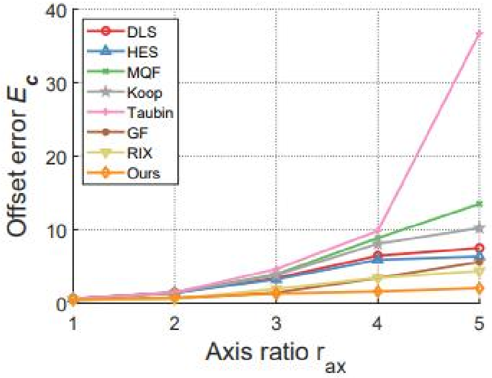


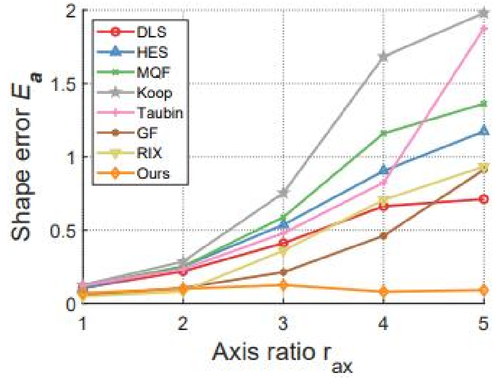
Our method is more robust against outliers

EXPERIMENTS— ROBUST AGAINST AXIS RATIO



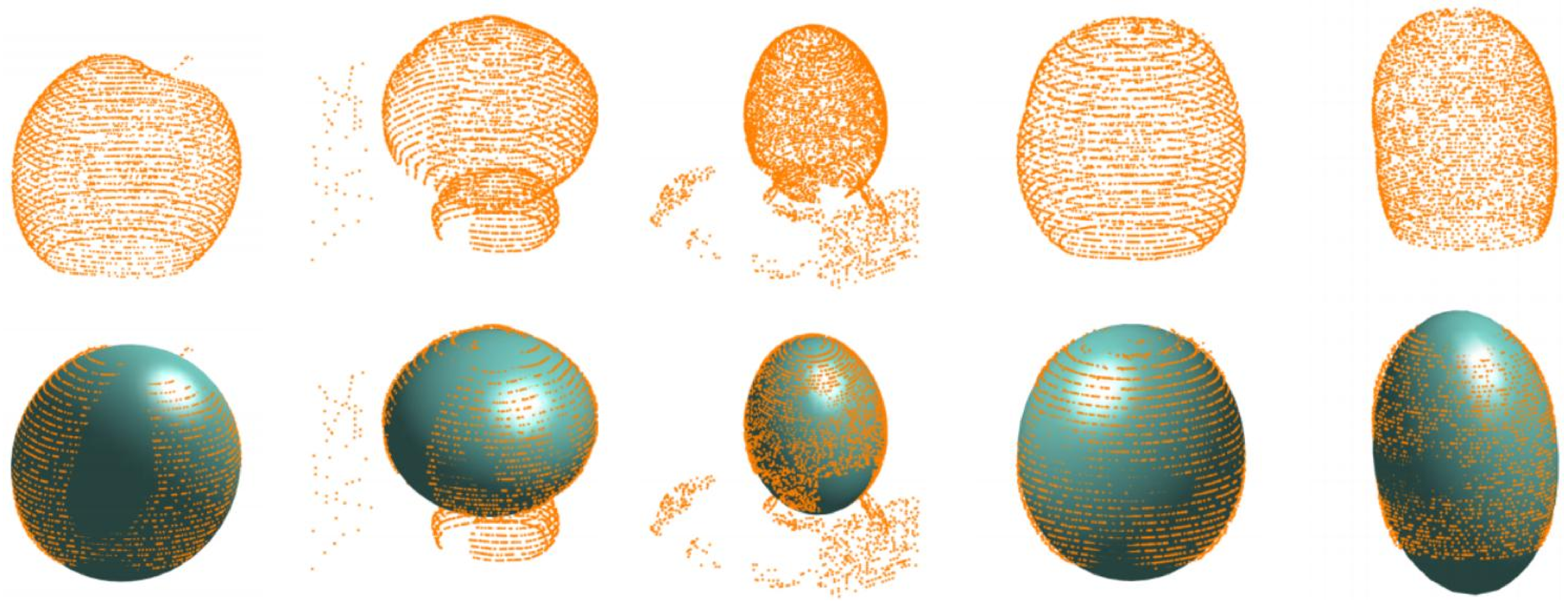
(a) DLS [23] (b) HES [22] (c) MQF [6] (d) Koop [39] (e) Taubin [37] (f) GF [4] (g) RIX [25] (h) Ours



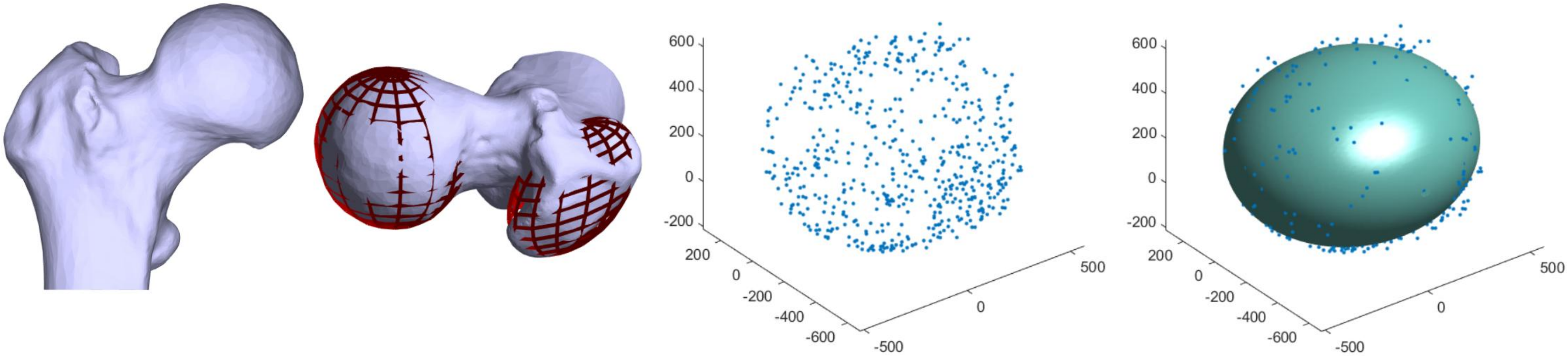


Our method is more robust against elongated ellipsoids

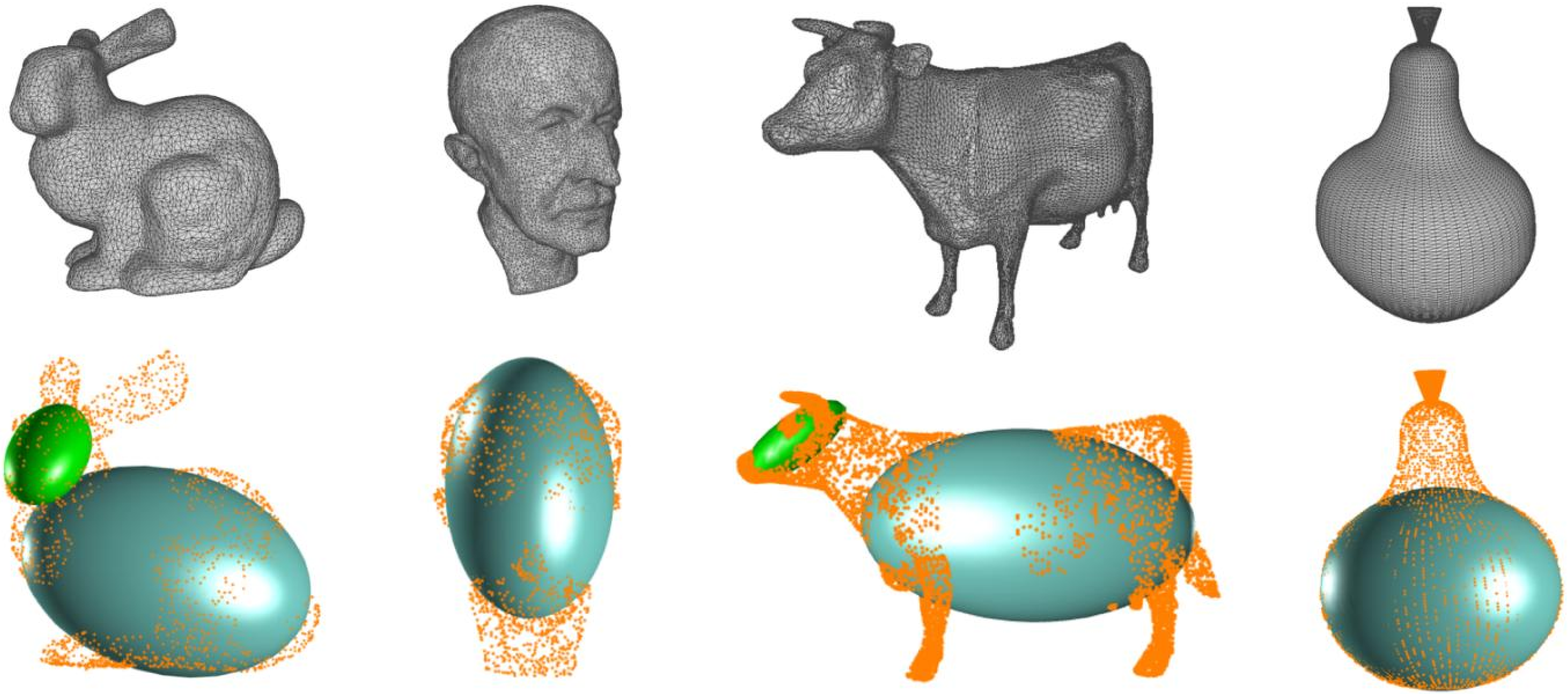
EXPERIMENTS— APPLICATION FOR POINT CLOUDS



Application of our method for real-world point clouds



Application of our method for medical data and magnetometer calibration



Application of our method for shape approximation

THANK YOU FOR YOUR ATTENTION!

Thank the anonymous reviewers for their valuable comments

Paper link: https://arxiv.org/abs/2110.13337

Code: https://zikai1.github.io/EllipsoidFit.